Chapter 3

Allometry in Biological Systems

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Introduction

The objectives of this exercise are (1) to understand the quantitative methods used to describe the relationship between one biological character and another, and (2) to examine experimentally the relationship between volume and surface area, and how this relationship affects heating and cooling rates. I have used this class successfully for the past 6 years in a sophomore-level vertebrate biology class. However, it is easily adapted to fit a more general class or a class with a different emphasis. The principles of allometry are the same whether one is talking about vertebrates, invertebrates, plants, or buildings. The exercise does not assume that the students have extensive math skills; the Prelab is written to guide even the most unquantitative student through the development of the allometric equation. This exercise also does not require access to a computer or statistical tools. However, it is easily adapted to include the use of computers if they are available. I include materials here to assist students in doing all calculations by hand; however, all of it is peripheral to the main focus of the exercise. This exercise takes less than 1 hour to set up, and about 2 hours to carry out if the students do the Prelab in advance of class. If you have to introduce all the material in class and guide them through the Prelab material, the entire exercise may take up to 4 hours.

Materials

Recommended material for each pair of students:

Beakers (4 sizes ranging from 250 ml to 2000 ml)
Graduated cylinders (4 sizes ranging from 250 ml to 2000 ml)
Immersion heaters (1)
Thermometers (4)
3 × 3 cycle log-log graph paper (10 sheets)
Arithmetic graph paper (10 sheets)
A clock that displays seconds (1)
Ruler (1)

Student Outline

Objectives

1. Understand the quantitative methods used to describe the relationship between one biological character and another.

2. Examine experimentally the relationship between volume and surface area, and how this relationship affects heating and cooling rates.

Comments to Students

It is important that you read this handout prior to coming to the laboratory and that you do the work associated with the Prelab Worksheet. Some of you learned this material in high school and can work through it quickly; however, don't assume that you know it. Many of you have never been exposed to allometry, worked with log-log paper, or calculated slopes of lines. It is especially important for you to do this work prior to coming to the laboratory. If you don't, you'll be completely lost during the exercise. You will find it convenient to bring a calculator that performs logarithms and exponents, and a wristwatch that displays seconds.

Prelab: Development of the Allometric Equation

Introduction to Allometry

As children, one of the first things we notice about an animal is its size. We say that elephants and whales are big, mice and frogs are small, and people are somewhere in between. As we grow older we realize that this apparent diversity of sizes is not a result of childish imagination: living organisms actually span a range of sizes from bacteria that weigh 10^{-10} g to blue whales that weigh in excess of 10^{8} g. Vertebrates cover only the top 40% or so of this range, from fish that weigh less than a gram to the blue whale.

Size differences are more than a simple way to classify organisms, however. Size is an important factor in evolution. The trend among most organisms has been for an increase in size over time; the first organisms seen in the fossil record are individual prokaryotic cells about 20 microns in diameter from 3400 million years ago (mya), and the largest organisms ever to have lived, the baleen whales, appeared about 34 mya. An increase in size is seen within many groups as well; the evolutionary history of horses, for example, shows a steady increase in size over 60 million years from the diminutive *Hyracotherium* to the modern *Equus*.

Reasons for this trend of increasing size are many. Larger size may be selected for because it allows an increase in individual complexity and regional specialization, an increased ability to find or process food, an advantage in acquiring mates or raising young, or an increased ability to avoid being eaten by predators.

Body size also plays an important role in the day-to-day operation of an organism. Everything about the biology of an animal is influenced by its body size, including its physiology (e.g., heart rate, respiratory rate, total metabolic rate, mass specific metabolic rate, growth rate), anatomy (e.g., organ mass, blood volume, surface area, cross-sectional area of limbs), and ecology (e.g., diet, home range size, reproductive strategy, life span, population density).

In short, one of the most general concepts that you are likely to find in biology is that of the relationship between body size and the rest of an organism's biology, a relationship that is often referred to as allometry. Therefore, an exploration of this topic is important if we are to understand the relationship between form and function in vertebrates. The purpose of this exercise is not to discuss all of the factors that are affected by body size but rather to examine the ways scientists talk about the relationship between body size and other biological factors, as well as to explore for ourselves the relationship between two important biological traits: volume (a function of mass) and surface area, and how this relationship influences other aspects of an animal's biology.

Quantitative Methods in Allometry

Allometry literally means "of other or different measures" (*allo* = other or different; *metry* = measure). The goal of its study with respect to biology is to describe the differences in magnitude in form or function that are correlated with changes in form or function of another variable. More simply, it is the study of what happens to Variable *Y* when you change Variable *X*.

This is basically a qualitative description and only allows us to speak of correlated changes in qualitative ways; for example, *Y* gets bigger as *X* gets bigger, *Y* slows down as *X* speeds up, *Y* gets wider as *X* gets heavier, and so on. As biologists, however, we want to be able to describe things in a more precise way. Does *Y* get bigger (or slow down or get wider) at the same rate as *X*? At a faster rate? A slower rate? To satisfy our need for quantification, we resort to mathematical equations that precisely describe the relationship between two variables. For example:

$$Y = X \tag{1}$$

is an equation that describes quantitatively how *Y* changes with *X*. It says that *Y* is always exactly the same as *X*. As *X* changes, *Y* changes in exactly the same way. Let's pretend, for example, that you and a friend are trying to fill a sandbox and that you are both shoveling sand at the same rate. If you shovel 1 kg of sand/minute, then your friend shovels 1 kg/minute. If you shovel 5 kg/minute, then your friend shovels 5 kg/minute. If you were to collect data on the relationship between the rate at which you shovel and the rate at which your friend shovels, your data set might look like that in Table 3.1A.

Table 3.1. The rate of shoveling of two people trying to fill a sandbox.

Rate of shoveling (kg/minute)					
A		B		C	
$Y = X$		$Y = 2X$		$Y = X^2$	
Person X	Person Y	Person X	Person Y	Person X	Person Y
1.0	1.0	1.0	2.0	1.0	1.0
2.0	2.0	2.0	4.0	2.0	4.0
3.0	3.0	3.0	6.0	3.0	9.0
5.0	5.0	5.0	10.0	5.0	25.0
10.0	10.0	10.0	20.0	10.0	100.0

In the Prelab Worksheet, plot these data on arithmetic (= standard) graph paper. Label the *X*-axis as Person X with the scale up to 10 and the *Y*-axis as Person Y with the scale up to 100. Connect the data points, and label the line $Y = X$ (Equation 1).

Equation 1 is not very general, however, because it can only describe systems in which the variables are always equal to one another. It does not describe systems in which (1) there is a relative difference between *X* and *Y*, and (2) the relative difference between *X* and *Y* is not constant. It would be useful, therefore, if we could modify Equation 1 in some way to allow us to describe these more complex situations. To understand how these modifications are made, let's look at these two situations separately.

Modification 1: There is a relative difference between X *and* Y

First, what is the relationship between two variables that have a constant relative difference. In our sandbox example, let's say that no matter how fast you shovel, your friend always shovels twice as fast as you. If you shovel 1 kg/minute, your friend shovels 2 kg/minute. If you shovel 5 kg/minute, your friend shovels 10 kg/minute. Your data set might look like that in the Table 3.1B.

Because *Y* is always twice that of *X*, the relationship between them can be described as:

$$Y = 2X \tag{2}$$

Plot these data on your first graph in the Prelab Worksheet, connect the data points, and label the line $Y = 2X$ (Equation 2).

Equation 2 is accurate for situations in which *Y* is always 2 times larger than *X*; however, we would like an equation that will be general enough to describe the relationship between *Y* and *X* no matter how much greater *Y* is than *X*. Such an equation is this:

$$Y = aX \tag{3}$$

where *a* is a constant that can equal any number. In our second example, in which you are shoveling sand at only half the rate of your friend, $a = 2$. What does *a* equal in our first example? Why do we call $Y = aX$ a general equation?

Modification 2: The relative difference between X *and* Y *is not constant*

The second situation we want to describe is more difficult. How can we quantify a system in which the relative difference between *X* and *Y* is not always the same? For example, what if you and your friend can both shovel at almost the same rate when you are not shoveling very fast (*X* almost equals *Y* when *X* and *Y* are small) but your friend can shovel a whole lot faster than you when you are both working quickly. The data may look like that in Table 3.1C.

Notice that when *X* and *Y* are small, there is no difference between them, and that as *X* gets bigger, the relative magnitude of *Y* gets bigger (i.e., twice as big, then three times as big, then five times as big, and so on). We need an equation that says that when *X* and *Y* are small, the relative difference between them is small and that as *X* and *Y* increase, the relative difference between them increases. The general equation $Y = aX$ won't help; this says that now matter what *X* is, *Y* is always *a* times greater than *X*.

For the situation described in Table 3.1C, we must resort to the use of exponents. In the case above, we can describe the relationship between the rate of shoveling by you and your friend as:

$$Y = X^2 \tag{4}$$

Equation 4 says that for every kg/minute you shovel, your friend shovels at that rate squared. Plot these data on your first graph in the Prelab Worksheet, connect the data points, and label the line $Y = X^2$ (Equation 4).

Equation 4 is not very general, however. Once again, we would like an equation that will describe the relationship between any two variables, no matter what the difference in the rate of change is. We can obtain this general equation by again using a variable that can assume any constant value; in this case, let's use the exponential constant *z*. Equation 4 then becomes:

$$Y = X^z \tag{5}$$

Equation 5 states that *Y* changes exponentially with respect to *X*, or that it is an exponential function of ***X*.** We sometimes refer to *z* as a scaling factor. Equation 5 is general because it can be used to describe the exponential relationship between *X* and *Y* no matter what the relationship is. If *Y* increases at a faster rate than *X*, then $z > 1$ (as in Table 3.1C and Equation 4). If *Y* increases as *X* increases, but doesn't increase as quickly, then $0 < z < 1$. If *Y* decreases as *X* increases, then $z < 0$. What is *z* when $Y = X$ (Equation 1)? What is *z* when *Y* remains constant?

The General Allometric Equation

We can now put these two general equations together into one equation that describes the relationship between any two variables:

$$Y = aX^z \tag{6}$$

When you plot Equation 6 on arithmetic graph paper, you will usually obtain some kind of curved line. Unfortunately, curved lines are often troublesome to describe, interpret, and subject to statistical analysis. To simplify the interpretation of the general allometric equation, two different techniques can be used. The first technique is complicated and rarely used, but I include it here so you can see why the second technique works.

The first technique is to convert the allometric equation to a form that is easier to analyze—a straight line. You may remember from your study of algebra that the equation for a straight line is:

$$Y' = mX' + b \tag{7}$$

where *m* equals the slope of the line (the ratio of the change in Y' with change in X', or “rise-over-run”) and *b* is the *Y*-intercept (the value of Y' when $X' = 0$). (Y' and X' are pronounced “Y prime” and “X prime.”)

Can we somehow convert the general allometric equation to this form? Yes. All we need to do is take the logarithm of both sides of Equation 6. To do this, you need to remember two algebraic rules (see Appendix A for their proofs):

1. To obtain the logarithm of the product of two numbers, take the log of each number separately and then add these values together. For example, the logarithm of 4 × 5 equals the log of 4 plus the log of 5. More generally, log *(ab)* = log *a* + log *b*.

2. To obtain the logarithm of a number raised to a power, multiply the exponent and the logarithm of the root. For example, the logarithm of 5^2 equals 2 times the log of 5. More generally, $\log a^b = b \log a$.

Using these two rules, you should be able to show that $Y = aX^z$ converts to:

$$\log Y = z \log X + \log a \tag{8}$$

Why is this of use? Remember what the equation for a line is (Equation 7). If the variables in Equation 8 are transformed as follows:

log *Y*	--->	*Y′*
z	--->	*m*
log *X*	--->	*X′*
log *a*	--->	*b*

then the log-transformed allometric equation becomes a linear equation. Prove this for yourself using the following equation:

$$Y = 2X^{1.7} \tag{9}$$

To do this proof, first plot the untransformed equation on arithmetic graph paper. (Appendix B may be of help to you.) Pick some values for *X* and calculate *Y*. Plot *X* vs. *Y*, connect the data points, and label the curve.

Next, log-transform Equation 9, pick values for *X*, calculate log *Y* (again, Appendix B may be of help), and plot log *X* vs. log *Y* on a new labelled graph. What is the shape of the line?

All of these transformations can be rather tedious and prone to error. Fortunately, there is another method whereby the general allometric equation can be made into a straight line. This technique takes advantage of the fact that the plots of equations can be log-transformed not only by transformation of the equation, but also by plotting the data on a log-transformed graph. Such a graph is called a log-log plot. Compare the *X*- and *Y*-axes on log-log graph paper (the second kind of graph paper included with the Prelab Worksheet) to those on arithmetic graph paper. Notice that on the arithmetic graph, *X* and *Y* values increase in an arithmetic progression. That is, the distance between lines is the same whether you are going from 4 to 5 or 400 to 401. On the log-log graph, however, they increase in a logarithmic progression. Each cycle of 1 to 10 on an axis actually represents the number 1 to 10 × 10 to some exponential power. For example, your *X*-axis could represent 1 to 10×10^2, 1 to 10×10^3, and 1 to 10×10^4. The exponential power used in the first cycle determines what powers are used in the second and third cycles; they are exactly 1 and 2 powers greater. Now the distance between 4 and 5 (4×10^0 and 5×10^0) is different than the distance between 400 and 401 (4×10^2 and 4.01×10^2). The same is true of the *Y*-axis. Two other things to note about log-log paper: (1) the exponents used for the *X*- and *Y*-axes do not need to be the same, and (2) you can never get to the point 0,0.

A log-log plot allows us to convert the general allometric equation to a straight line without having to take the logarithms of any numbers! This is useful when you don't have access to a log table, calculator, or computer. Show that an equation of the general allometric form on log-log paper results in a straight line by graphing and labelling Equation 9.

The applicability of this allometric equation is one of the most general tools we have in the biological sciences. It allows us to explain how any physiological, morphological, or ecological variable changes with any other variable. In actual application, biologists always just plot their allometric data on log-log paper. The resulting line is straight and easy to describe or compare.

Going from Data to Equations

Up to now we have talked as if generating an allometric equation from data was straightforward. In Table 3.1, the data were so "perfect" that they could be plotted on a graph and a line drawn through them with great accuracy. In real life, however, data are not like that. How then do you fit a line to the data and calculate the allometric equation? There are two ways.

The first is simply to draw a line by eye through the data points such that about half of the points appear to lie above the line and half appear to lie below the line. From there, you can calculate the slope and estimate the *Y*-intercept (remember that on log-log paper you can never reach zero!). This technique may do in a pinch but it is not very accurate if you need to compare two allometric equations or try to understand the exact relationship between two characters.

The second way involves using some statistical technique, usually a linear regression, on the data. This can be easily done with a calculator or computer that has a statistics package. The regression equation describes the average relationship of *X* and *Y*, which can then be used to generate a line. If you don't have access to a linear regression program, you can calculate the regression equation by hand using the method described below and demonstrated in Appendix C.

The procedure for calculating the allometric equation from a set of data is as follows: Take the log of both your *X* and *Y* values and then calculate the regression equation for log *X* and log *Y*. Why do you need to take the logarithms of your *X* and *Y* values? Because the regression gives you a linear equation, and you want to be able to transform it into an allometric equation. It's the reverse of what we did to go from Equation 6 to Equation 8. The slope of the equation equals *z* and the antilog of the *Y*-intercept equals *a*. Insert the values for *z* and *a* into the general allometric equation.

Calculate the allometric equation for the data in Table 3.2. To calculate the equation you may use a computer, programmable calculator, or the method in Appendix C.

Once the allometric equation is generated from your data, you can use it to describe how *Y* changes with *X*, what proportion of a change in *Y* is associated with a change in *X*, or what data points are far from average (i.e., far from the allometric line) and deserve special attention.

Table 3.2. Data set to practice performing a linear regression.

X	*Y*
1.1	10.9
8.0	18.0
10.2	20.1
14.9	20.8
15.6	23.1
32.4	26.0
46.5	27.5

Limitations of Allometric Equations

Before we begin our experiments, you should be aware of two limitations of allometry. The first is that an allometric equation only describes the relationship between two characters; it does not explain why the relationship is the way it is. An understanding of the basis for a particular relationship can only come from knowledge of the system itself and may not be obvious. For example, the relationship between body mass and basal metabolic rate in passerine birds is:

$$\text{kcal/day} = 78.3(\text{kg})^{0.72}$$

This tells us that metabolic rate increases less quickly than does body mass and it allows us to predict the metabolic rate of a bird of known mass. However, it does not tell us why metabolic rate increases less quickly than mass. We don't really know the answer to that. It may be related to the allometric relationship between body mass and surface area, body mass and active tissue mass, or something else altogether.

The second limitation is that the relationship between two traits may change over time and that one allometric equation is insufficient to describe it accurately. For example, early in the life of a lizard the relationship between age and growth may be:

$$\text{mg/day} = 3.63(\text{days})^{0.88}$$

while after adulthood it may be:

$$\text{mg/day} = 3.63(\text{days})^{0.40}$$

One equation that describes the relationship between age and growth rate over the entire life of an animal may be a good average but not descriptive for any particular range of ages. You must inspect your data before deciding how to describe the relationship between two characters. For further reading see Alexander (1985), Calder (1984), McMahon and Bonner (1983), Peters (1983), Schmidt-Nielsen (1984), and Vogel (1988).

Laboratory Exercise

Surface Area to Volume (SA:V) Ratios

Work in pairs to determine the relationship between the surface area and the volume of "organisms" of different size. There may not be enough of all the equipment for every pair to have their own complete set so some equipment will have to be shared. You may use the data sheets available at the front of the class to record your data in an organized fashion.

The first set of organisms in this experiment are glass beakers. Calculate the allometric equation for the SA:V ratio for beakers. Plot your data on log-log paper (volume should always be on the *X*-axis) and record the allometric equation on the Laboratory Worksheet.

Next, perform the same analysis with a different class of organisms, that of graduated cylinders. Plot the values on the same log-log plot in the Laboratory Worksheet and calculate the allometric equation for SA:V for cylinders. Answer the questions as directed on the Laboratory Worksheet.

Physiological Consequences of Body Size

Perhaps the greatest influence on animal physiology is body size. Using an immersion heater to simulate metabolic heat production by beakers, calculate (1) the relationship between body mass (= volume) and heating rate, and (2) the relationship between body mass (= volume) and cooling rate. First, a word of warning. Never plug in the heater without first placing it in the water and never remove the heater from the water without first unplugging it. Failure to follow these guidelines will result in permanent damage to the heater. You may use the data sheets available at the front of the class to record your data in an organized fashion. For each of the four beakers, and starting with the largest beaker, follow these procedures:

1. Fill the beaker with the same volume of water that you used in the first exercise. It is best if the water starts off at a temperature near that of the room.

2. Place the unplugged heater in the water and allow the temperature of the water to equilibrate.

3. Plug in the heater and, at the same moment, record the exact time and the water temperature.

4. Allow the water to rise about 20°C (the actual amount of the increase is not important as long as you record it precisely). Swirl the water continuously with the thermometer to obtain complete mixing and accurate temperature measurements.

5. Unplug the heater, remove it from the water and, at the same moment, record the exact time and water temperature. Use these data to calculate the heating rate in °C/second.

6. Use the ending time and temperature for the heating trial as the starting time and temperature of the cooling trial. Allow the water to drop about 5°C. (*Warning:* this will take a lot longer than the heating). Record the exact time and water temperature at the end of your trial. Use these data to calculate the cooling rate in °C/second.

7. Pour out the water and repeat this procedure twice more.

8. Repeat steps 1–7 for all four sizes of beakers. In actual practice, you should run several trials concurrently because cooling takes so much longer than heating

9. Plot the data for volume vs. heating rate on log-log paper, calculate the allometric equation that describes those data, and draw the line on the graph. Repeat this for volume vs. cooling rate. You may use two pieces of graph paper if the range of values on the *Y*-axes do not overlap. Use these data to answer the questions as directed on the Laboratory Worksheet.

Prelab Worksheet

General Instructions: Use the graph paper provided to plot your data. You must clearly label all graphs, axes, axes units, and lines. You must turn in this worksheet at the start of the laboratory exercise. You may write your answers on a separate sheet if you clearly identify the question to which your answer responds.

1. Plot Equations 1, 2, and 4 on arithmetic graph paper.

   (a) What is the shape of Equation 1?
   (b) What is the shape of Equation 2?
   (c) How does Equation 2 compare to Equation 1?
   (d) What is the shape of Equation 4?
   (e) How do exponential and non-exponential equations differ?
   (f) Why is Equation 6 a general equation?
   (g) When will the general allometric equation result in a straight line?

2. Plot Equation 9 on arithmetic graph paper. Plot the log-transformed version of Equation 9 on a different arithmetic graph.

   (a) How do these two plots compare?
   (b) What is the slope of the log-transformed equation?
   (c) What is its *Y*-intercept?

3. Plot Equation 9 on log-log graph paper.

   (a) How does this compare to your plot of the same equation on arithmetic paper?
   (b) Are there any values of *a* and *z* in the general equation that do not result in a straight line when plotted on log-log paper?

4. Plot the data in Table 3.2 on log-log paper. Calculate the allometric equation that best describes the data and draw in the line that represents that equation.

Laboratory Worksheet

General Instructions: Use the graph paper provided to plot your data. You must clearly label all graphs, axes, axes units, and lines. You may write your answers on a separate piece of paper but only if you clearly identify the question to which your answer responds.

1. Plot your data for surface area and volume for both beakers and graduated cylinders on the same log-log plot. Calculate the allometric equation for each data set separately and draw and label the lines described by those equations.

   (a) What is the allometric equation for beakers?
   (b) What is the allometric equation for graduated cylinders?
   (c) Can you determine theoretically why z is roughly equal to 0.67?

2. (a) How do beakers differ from cylinders?
   (b) For a given shape, what size has the largest SA:V ratio?
   (c) How does shape influence SA:V ratios?

3. Plot your data for the heating and cooling rates of beakers on log-log plots. Calculate the allometric equation for each data set and draw and label the lines described by those equations.

   (a) What is the allometric equation for the heating rate of beakers?
   (b) What is the allometric equation for the cooling rate of beakers?
   (c) What is the relationship between heating rate and body size?
   (d) What is the relationship between cooling rate and body size?

4. From the data in Table 3.3, one author drew the following conclusion: "Since tree shrews have lower metabolic rates than insectivores and higher metabolic rates than primates, the placement of the tree shrews in an intermediate order, the Menotyphia, by some paleontologists is supported by a physiological parameter." Given the following data on body mass of these species (shrew = 3.5 g; tree shrew = 12.5 g; chimpanzee = 48.0 kg; human = 70.0 kg), provide an alternative conclusion via allometry. (This question is adapted from Bill Calder's book *Size, Function, and Life History*).

Table 3.3. Comparative metabolic rates for selected mammals.

Species	Metabolic Rate cal/(kg × hour)
Shrew	66.7
Tree shrew	10.0
Chimpanzee	3.5
Human	1.2

5. (a) If a 20,000 kg dinosaur were to warm its body to 40°C in the sun before sunset, how many degrees would its body temperature drop in a 12-hour night? Assume that dinosaurs are shaped like beakers and that 1 g of dinosaur equals 1 cc of water.

   (b) If dinosaurs can only tolerate having their body temperature drop to 30°C, what is the smallest dinosaur that could survive a 12-hour night?

   (c) What does this tell you about the relative abilities of large and small dinosaurs to stay warm at night without having to generate heat metabolically? (Hint for solving these problems: Use your equation for cooling rate, plug in the known variables, make sure the units are correct, and solve for the unknown.)

Notes for the Instructor

Upon request, I can provide a copy of this handout and most support material on a 3.5" disk as a Word 4.0 file for the Macintosh, as an Interchange (RTF) file readable by Word 5.0 for the IBM, or as a simple text file.

Some students find this exercise frustrating at first because they do not take the time to read the handout thoroughly or to work carefully through the exercises. Anything that you can do to instill care and precision in their work will greatly improve their enjoyment of and success with this exercise.

I have the students do the Prelab on their own time. If you do that, be sure that they obtain this handout and several sheets of both types of graph paper at least 1 week before the lab session.

Over the years I have discovered that no amount of pleading gets all of the students to do the Prelab before the start of class. As a result, during the laboratory there are groups of students doing a wide range of things, from trying to finish the Prelab to working quickly through the exercise. This can be a difficult situation for an instructor without an assistant. Therefore, you may decide to do the work described here in two class sessions, the first covering the material in the Prelab and the second covering the formal exercise.

This exercise works best if the students work in groups of two. Each group should have access to beakers of four different sizes that span a range of at least two orders of magnitude. The smallest should be about 100 ml and the largest around 2000 ml. Each pair should also have access to four different cylinders that span the same range of sizes as the beakers, although the exact sizes do not need to be the same. Some of the very largest cylinders (e.g., 2000 ml) can be expensive and hard to obtain; in these cases, fewer than one per group can be provided and groups can share them.

Immersion heaters are heating coils used to heat up individual cups of water; they should be available in most hardware stores for around $5 US each.

The calculations in this exercise involve logarithms, antilogs, exponents, and regression equations. I encourage students to use any device at their disposal, including calculators and statistical packages on micro- or mainframe computers. I use primes in Equation 7 to avoid confusion with the Y and X in the general allometric equation.

Thermometers should record between at least 0° and 100°C. Each group can get by with only one, but it is more efficient if they have four of them. The students can also use the thermometers as stirring rods.

Rulers must be able to measure the height of the tallest cylinder. I use 1-m tape measures for this. Stacks of many sheets of 3 × 3 cycle log-log graph paper and arithmetic graph paper should be available in class as well as provided with the Prelab Worksheet. If the laboratory room is not equipped with a clock with a sweep second hand, then be sure that each group has a wristwatch that displays seconds.

I tend to give very little guidance to the students in the Laboratory Exercise and let them work in their groups to figure out what they have to do through the "ah-ha" method. You may choose to demonstrate to them exactly what they have to do.

Here is some useful information to have for the Laboratory Exercise:

1. $\text{SA} = (\text{height} \times \text{diameter} \times \text{ð}) + 2(\text{ð} \times (\text{diameter}/2)^2)$

2. $\text{V} = \text{height} \times (\text{diameter}/2)^2 \times \text{ð}$, or it can be read off the gradations on the side

3. Beakers and cylinders have irregularly shaped lips; have the students choose a volume below the lip and calculate the SA of that volume. For example, on a 250 ml beaker, choose th volume to be 200 ml and then calculate the SA of the beaker only to the height of the 200-ml mark.

4. Use the inside measurements of the beakers and cylinders.

5. Make sure that the units of measure are kept constant.

Appendices D and E provide the answers to the questions in the Prelab Worksheet and to the additional questions that were asked in the Prelab. The answers provided in Appendix F are merely representative of what the students will come up with; each group will have slightly different values. Blank data sheets are provided in Appendices G and H.

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APPENDIX A
Algebraic Proofs

1. Proof that $\log (ab) = \log a + \log b$

Set: $a = 10^x$ and $b = 10^y$

It follows by definition that: $\log a = x$ and $\log b = y$

Therefore, by definition and substitution:

$$\begin{aligned} \log (ab) &= \log (10^x \times 10^y) \\ &= \log (10^{x+y}) \\ &= x + y \\ &= \log a + \log b \end{aligned}$$

2. Proof that $\log a^b = b \log a$

Set: $a = 10^x$ and $b = y$

It follows by definition that: $\log a = x$

Therefore, by definition and substitution:

$$\begin{aligned} \log a^b &= \log (10^x)^y \\ &= \log 10^{x \times y} \\ &= x \times y \\ &= y \times x \\ &= b \log a \end{aligned}$$

APPENDIX B
Conversions for Graphing Equation 7

X	$X^{1.7}$	$\log X$
1	1.0	0
2	3.3	0.30
5	15.4	0.70
10	50.1	1.00
20	162.8	1.30
40	529.0	1.60
100	2511.9	2.00

APPENDIX C
Calculation of Allometric Equation

This is a demonstration of how to calculate the allometric equation by hand, using linear regression and reverse log-transformation, with four data points (n = 4). List your paired values of X and Y. Convert these to log X and log Y, which we abbreviate to X' and Y' as in Equation 8. The calculations following that are for linear regression. Note: you may obtain a slightly different answer if you use more than two significant digits. The table below provides a spreadsheet of the data and initial conversions to demonstrate the calculation of linear regression.

X	Y	$X' = \log X$	$Y' = \log Y$	$X'Y'$	$q = (X - X')$	q^2
100	15.5	2.00	1.19	2.38	0.52	0.27
250	4.3	2.40	0.63	1.51	0.12	0.01
500	2.1	2.70	0.32	0.86	-0.18	0.03
1000	1.2	3.00	0.08	0.24	-0.48	0.23
Σ		10.10	2.22	4.99		0.55
Mean		2.52	0.56			

$s = \Sigma X'Y' - ((\Sigma X' \times \Sigma Y')/n) = 4.99 - ((10.10 \times 2.22)/4) = 4.99 - 5.61 = -0.62$

$\text{slope} = s / \Sigma q^2 = -0.62/ 0.55 = -1.13$

$Y\text{-intercept} = \overline{Y'} - (\text{slope} \times \overline{X'}) = 0.56 - (-1.13 \times 2.52) = 0.56 + 2.84 = 3.40$

The linear equation is therefore $Y' = -1.13\, X' + 3.40$. The allometric equation is obtained by taking the inverse logarithm of the linear equation. Remember that $Y' = \log Y$ and $X' = \log X$. Therefore:

$\log Y = -1.13 (\log X) + 3.40$

$Y = (\text{antilog } 3.40)\, X^{-1.13}$

$Y = 2512\, X^{-1.13}$

APPENDIX D
Answers to the Prelab Worksheet

1. Plot of Equations 1, 2, and 4 on arithmetic graph paper:

(a) Straight line.
(b) Straight line.
(c) It has twice the slope.
(d) Curve; parabola.
(e) Non-exponential equations are straight lines on arithmetic graph paper, whereas exponential equations are usually curved.
(f) Because it can describe the relationship between X and Y no matter what the relationship is.
(g) When z = 0 or z = 1.

2. Plot of Equation 9 on arithmetic graph paper and plot of the log-transformed version of Equation 9 on a different arithmetic graph:

(a) The first one is curved and the second is straight.
(b) 1.7
(c) 0.3

3. Plot of Equation 9 on log-log graph paper:

(a) This time it is straight.
(b) No.

4. Plot of the data in Table 3.2 on log-log paper: $Y = 10.91\ X^{0.25}$

APPENDIX E
Answers to Additional Prelab Questions

1. What does a equal in Equation 1? 1

2. Why do we call $Y = aX$ a general equation? Because no variable is specified and all of them can take on any value.

3. What is z when $Y = X$? 1

4. What is z when Y remains constant? 0

5. What is the shape of the log-transformed Equation 9? Straight

Proof of derivation of Equation 8: $Y = aX^z$

$\log Y = (\log a) + \log (X^z) = \log a + z \log X = z \log X + a$

Log-transformation of Equation 9: $Y = 2X^{1.7}$

$\log Y = (\log 2) + \log (X^{1.7}) = \log 2 + 1.7 \log X = 1.7 \log X + 0.3$

APPENDIX F
Answers to the Laboratory Worksheet

1. (a) $Y(\text{cm}^2) = 4.2\ X(\text{cm}^3)^{0.65}$
   (b) $Y(\text{cm}^2) = 11.4\ X(\text{cm}^3)^{0.66}$
   (c) By the following proof, with Length = L:
   SA is proportional to (α) L^2 and $V\ \alpha\ L^3$
   $SA^{1/2}\ \alpha\ L$ and $V^{1/3}\ \alpha\ L$
   $SA^{1/2}\ \alpha\ V^{1/3}$
   $SA\ \alpha\ V^{2/3}$

2. (a) There is less surface area at any given volume.
   (b) Smallest size.
   (c) The more elongated or flattened the shape, the more SA at any given volume.

3. Plot of data for the heating and cooling rates of beakers on log-log plots:

   (a) $Y(°\text{C/second}) = 21.63\ X(\text{ml})^{-0.85}$
   (b) $Y(°\text{C/second}) = 0.80\ X(\text{ml})^{-0.82}$
   (c) Heating rate decreases with increasing body size.
   (d) Cooling rate decreases with body size; also, at any given body size the cooling rate is less than the heating rate.

4. The metabolic rate of tree shrews is relatively what you would predict if metabolic rate were controlled solely by body mass. This can be demonstrated by calculating the allometric equation of body mass vs. metabolic rate for shrews, chimps, and humans [Y (cal/(kg × hour)) = 104.7 $X(\text{g})^{-0.36}$] and calculating the predicted metabolic rate for tree shrews (42.1 cal/(kg × hour).

5. (a) 20,000 kg = 20,000,000 cc
   12 hours = 43200 second
   $Y(°\text{C/second}) = 0.80\ X(\text{ml})^{-0.82}$
   Answer = $(0.80)((2 \times 10^7)^{-0.82}) \times 43200$ second = 3.5×10^{-2}°C

   (b) 40 - 30 °C/43200 second = $0.80\ X^{-0.82}$
   2.3×10^{-4} °C/second = $0.80\ X^{-0.82}$
   $2.8 \times 10^{-4} = X^{-0.82}$
   $\log(2.8 \times 10^{-4}) = -0.82 (\log X)$
   $4.32 = \log X$
   $2.06 \times 10^4 = X$
   Answer: 20.6 kg

(c) Large dinosaurs are able to maintain a relatively constant body temperature even at night because their large mass results in such incredibly small cooling rates. Small dinosaurs would have to generate more heat metabolically to maintain a constant body temperature at night.

APPENDIX G
SA:V Ratio Data Sheet

Beaker size	Height (cm)	Diameter (cm)	Volume (cm^3)	Surface Area (cm^2)

Cylinder size	Height (cm)	Diameter (cm)	Volume (cm^3)	Surface Area (cm^2)

APPENDIX H

Heating and Cooling Rate Data Sheet

Heating rate								
Volume	Trial	Staring		Ending		δ Temp.	δ Time	Rate
		Temp.	Time	Temp.	Time			

Cooling rate								
Volume	Trial	Staring		Ending		δ Temp.	δ Time	Rate
		Temp.	Time	Temp.	Time			